

# HISTORIC BEHAVIOUR FOR QUENCHED RANDOM EXPANDING MAPS ON THE CIRCLE

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**ABSTRACT.** Takens constructed a residual subset of the state space consisting of initial points with historic behaviour for expanding maps on the circle. We prove that this statistical property of expanding maps on the circle is preserved under small quenched random perturbations. The proof is given by establishing a random Markov partition, which follows from a random version of Shub's Theorem on topological conjugacy with the folding maps. As a by-product, we obtain a new formula for the unique absolutely continuous ergodic invariant probability measure of random expanding maps on the circle.

## 1. INTRODUCTION

Let  $M$  be a compact smooth Riemannian manifold. For a dynamical system  $f : M \rightarrow M$ , the orbit issued from an initial point  $x$  in  $M$  is said to have historic behaviour when there exists a continuous function  $\varphi : M \rightarrow \mathbb{R}$  such that the time average

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi(f^j(x))$$

does not exist. Several statistical quantities are given as the time average of some observable  $\varphi$ , and thus it is a natural question in smooth dynamical systems theory whether the set of points with historic behaviour is not negligible in some sense; it is typically formulated as a positive measure set with respect to the normalised Lebesgue measure. Takens conjectured in [11] that there are persistent classes of smooth dynamical systems such that the set of initial points with historic behaviour is of positive Lebesgue measure, but this problem is still open. The reader is asked to see [5, 10, 11] for the background of historic behaviour in this measure-theoretical sense; see also the (recent) significant progress to Takens' conjecture by Hofbauer and Keller [7], and by Kiriki and Soma [9].

As another measurement to investigate historic behaviour, Takens [11] considered whether the set of points with historic behaviour is a residual subset of the state space (i.e., the set of points with historic behaviour is not negligible in a topological sense). For the doubling map on the circle, he constructed a residual subset consisting of initial points with historic behaviour by using symbolic dynamics. Since the method of symbolic dynamics is applicable to any expanding maps on the circle, the proof can be literally translated to expanding maps on the circle (see also [4] for another investigation of historic behaviour for expanding maps). Our goal in this paper is to extend his result for historic behaviour (and its generalisation to expanding maps on the circle) to a quenched random setting. For

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a random version of Takens' conjecture in measure-theoretical sense, we refer to the result by Araújo [1]; in contrast to our result in topological setting, he gave a negative answer to a random version of Takens' conjecture in measure-theoretical sense for general diffeomorphisms under some conditions on noise.

As in the unperturbed case, the key step in the proof is establishing a (random) Markov partition. This is given through proving a random version of Shub's Theorem on topological conjugacy of expanding maps with the folding maps. As a by-product of this extension we will give a formula for (the density function of) the unique absolutely continuous ergodic invariant probability measure of random expanding maps, which is a new formula to the best of our knowledge.

**1.1. Definitions and results.** Let  $\mathbb{S}^1$  be the unit circle given by  $\mathbb{S}^1 = \mathbb{R}/\mathbb{Z}$ . Let  $\mathcal{C}^r(\mathbb{S}^1, \mathbb{S}^1)$  and  $\text{Homeo}(\mathbb{S}^1, \mathbb{S}^1)$  be the spaces of all endmorphisms of class  $\mathcal{C}^r$  and homeomorphisms on the circle  $\mathbb{S}^1$ , endowed with the usual  $\mathcal{C}^r$  and  $\mathcal{C}^0$  metrics  $d_{\mathcal{C}^r}(\cdot, \cdot)$  and  $d_{\mathcal{C}^0}(\cdot, \cdot)$ , respectively, with  $r > 1$ . (Given that  $r = k + \gamma$  for some  $k \in \mathbb{N}$ ,  $k \geq 1$  and  $0 \leq \gamma \leq 1$ ,  $f \in \mathcal{C}^r(\mathbb{S}^1, \mathbb{S}^1)$  denotes the  $k$ -th derivative of  $f$  is  $\gamma$ -Hölder.) Let  $\mathcal{F}(\mathbb{S}^1)$  be the space of all nonempty left-closed and right-open intervals and all point sets of  $\mathbb{S}^1$ , endowed with the Hausdorff metric  $d_H(\cdot, \cdot)$ . We endow  $\mathcal{C}^r(\mathbb{S}^1, \mathbb{S}^1)$ ,  $\text{Homeo}(\mathbb{S}^1, \mathbb{S}^1)$  and  $\mathcal{F}(\mathbb{S}^1)$  with the Borel  $\sigma$ -field.

Let  $\Omega$  be a separable complete metric space endowed with the Borel  $\sigma$ -field  $\mathcal{B}(\Omega)$  with a probability measure  $\mathbb{P}$ . Given a smooth map  $f_0 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  of class  $\mathcal{C}^r$ , let  $\{f_\epsilon\}_{\epsilon > 0}$  be a family of continuous mappings defined on  $\Omega$  with values in  $\mathcal{C}^r(\mathbb{S}^1, \mathbb{S}^1)$  such that

$$(1.1) \quad \sup_{\omega \in \Omega} d_{\mathcal{C}^r}(f_\epsilon(\omega), f_0) \rightarrow 0 \quad \text{as } \epsilon \rightarrow 0.$$

For each  $\epsilon > 0$ , adopting the notation  $f_\epsilon(\omega, \cdot) = f_\epsilon(\omega)$ , the distance between  $f_\epsilon(\omega, x)$  and  $f_\epsilon(\omega', x)$  is bounded by  $d_{\mathcal{C}^r}(f_\epsilon(\omega), f_\epsilon(\omega'))$  for each  $x \in \mathbb{S}^1$  and each  $\omega, \omega' \in \Omega$ . Thus, it is straightforward to realize that  $f_\epsilon : \Omega \times \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is a continuous (in particular, measurable) mapping. When convenient, we will identify  $f_0 : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  with the constant map  $\Omega \ni \omega \mapsto f_0$ .

Let  $f_0$  be an expanding map on the circle, i.e., there exists a constant  $\lambda_0 > 1$  such that  $\inf_x \left| \frac{d}{dx} f_0(x) \right| \geq \lambda_0$ . For the properties of expanding maps, the reader is referred to [8]. Let  $k \geq 2$  be the degree of the covering map  $f_0$ . In view of (1.1) it then follows that  $f_\epsilon(\omega)$  is an expanding map for each  $\omega \in \Omega$  if  $\epsilon$  is sufficiently small. In fact, if  $\lambda_0 = \inf_x \left| \frac{d}{dx} f_0(x) \right|$  and we set  $\lambda = (\lambda_0 + 1)/2$ , then  $\lambda > 1$  and we can find an  $\epsilon_0 > 0$  such that

$$(1.2) \quad \inf_{\omega} \inf_x \left| \frac{\partial}{\partial x} f_\epsilon(\omega, x) \right| \geq \lambda, \quad 0 \leq \epsilon < \epsilon_0.$$

Let  $\delta_0 < \frac{1}{2}(1 - \frac{1}{k})$  be a positive number and  $\eta$  a positive number such that

$$(1.3) \quad \eta < \min \{1, (\lambda - 1)\delta_0\}.$$

We also assume  $\epsilon_0$  to be sufficiently small such that

$$(1.4) \quad \sup_{\omega \in \Omega} d_{\mathcal{C}^0}(f_\epsilon(\omega), f_0) < \frac{\eta}{2}, \quad 0 \leq \epsilon < \epsilon_0.$$

In particular,  $f_\epsilon(\omega)$  is the covering map of  $\mathbb{S}^1$  with degree  $k$  for each  $\omega \in \Omega$ .

Since  $f_0$  is the covering of  $\mathbb{S}^1$  of degree  $k \geq 2$ , there exists a fixed point  $p_0 \in \mathbb{S}^1$  for  $f_0$ . In view of (1.1) and (1.2) together with that  $f_0$  is locally a diffeomorphism of the circle, we can find a closed interval  $B = B_\epsilon$  including  $p_0$  such that if  $0 \leq \epsilon < \epsilon_0$ ,

then  $f_\epsilon(\omega) : B \rightarrow f_\epsilon(\omega)(B)$  is a diffeomorphism such that  $B \subset f_\epsilon(\omega)(B)$  for each  $\omega \in \Omega$  (by taking  $\epsilon_0$  sufficiently small if necessary). We assume that  $\epsilon_0$  is sufficiently small such that

$$(1.5) \quad \text{diam}(B) \leq \delta_0$$

for all  $0 \leq \epsilon < \epsilon_0$ , where  $\text{diam}(B)$  is the diameter of  $D$  with respect to the usual distance of  $\mathbb{S}^1$ . From the condition (1.5), it follows that there exists a point which is in  $\mathbb{S}^1$  but not in  $B$ . For simplicity we translate the point to 0, so that the distance between  $x, y \in B$  with respect to the usual distance of  $\mathbb{S}^1$  coincides with  $|x - y|$ .

*Remark.* When there is no ambiguity, the noise level  $\epsilon$  will sometimes be omitted from the notation, in particular when the dependence on the noise parameter  $\omega \in \Omega$  is already displayed. Throughout the rest of the paper we will also permit us to use  $\epsilon_0$  as a way to denote the upper bound of a range  $0 \leq \epsilon < \epsilon_0$  for which (1.2), (1.4) and (1.5) hold, even if  $\epsilon_0$  may change between occurrences. This will basically be showcased only in Theorem 11.

Let  $f : \Omega \rightarrow \mathcal{C}^r(\mathbb{S}^1, \mathbb{S}^1)$  be a measurable mapping, and  $\theta : \Omega \rightarrow \Omega$  a measure-preserving homeomorphism on  $(\Omega, \mathbb{P})$  (see the remark below the proof of Theorem 5 for this condition). For simplicity, we also assume  $\theta$  to be ergodic. For each  $n \geq 1$ , let  $f^{(n)}(\omega, x)$  be the fibre component in the  $n$ -th iteration of the skew product mapping

$$\Theta(\omega, x) = (\theta\omega, f(\omega, x)), \quad (\omega, x) \in \Omega \times \mathbb{S}^1,$$

where we simply write  $\theta\omega$  for  $\theta(\omega)$ . Setting the notation  $f_\omega = f(\omega, \cdot)$  and  $f_\omega^{(n)} = f^{(n)}(\omega, \cdot)$ , the explicit form of  $f_\omega^{(n)}$  is

$$f_\omega^{(n)} = f_{\theta^{n-1}\omega} \circ f_{\theta^{n-2}\omega} \circ \cdots \circ f_\omega.$$

For convenience, we set  $f_\omega^{(0)} = \text{id}_{\mathbb{S}^1}$  for each  $\omega \in \Omega$ . We call  $\{f_\omega^{(n)}(x)\}_{n \geq 0} = \{x, f_\omega(x), f_\omega^{(2)}(x), \dots\}$  the *random orbit* of  $f$  issued from  $(\omega, x) \in \Omega \times \mathbb{S}^1$ . As in the treatment of the unperturbed case undertaken by Takens [11], we now define historic behaviour for random orbits.

**Definition 1.** We say that a random orbit issued from  $(\omega, x)$  has *historic behaviour* if the empirical measure

$$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f_\omega^{(j)}(x)}$$

does not converge to any probability measure on the circle in weak\* sense, where  $\delta_x$  is the Dirac measure at  $x$ .

Our goal is to prove the following theorem.

**Theorem 2.** *Let  $0 \leq \epsilon < \epsilon_0$ . For  $\mathbb{P}$ -almost every  $\omega \in \Omega$ , we can find a residual subset  $\mathcal{R}^\omega$  of  $\mathbb{S}^1$  such that for any  $x \in \mathcal{R}^\omega$  the random orbit of  $f_\epsilon$  issued from  $(\omega, x)$  has historic behaviour.*

## 2. THE PROOF

We start the proof by considering a random version of Dowker's Theorem. This theorem asserts that if we find a dense orbit with historic behaviour (of the unperturbed dynamics), then there exists a residual subset of the phase space such

that the orbit of each point in the set has historic behaviour. For this purpose, we need to give stronger forms of the definitions of dense orbit and historic behaviour for random orbits. When there is no confusion, we employ the notation  $f_\omega^{(n)} = f^{(n)}(\omega, \cdot)$  for  $f = f_\epsilon$  once  $0 \leq \epsilon < \epsilon_0$  is given. For a continuous function  $\varphi : \mathbb{S}^1 \rightarrow \mathbb{R}$ ,  $\omega \in \Omega$ ,  $x \in \mathbb{S}^1$  and  $n \geq 1$ , we define the truncated time average  $B_n(\varphi; \omega, x)$  of the observable  $\varphi$  by

$$B_n(\varphi; \omega, x) = \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f_\omega^{(j)}(x).$$

Note that the orbit issued from  $(\omega, x)$  has historic behaviour if  $B_n(\varphi; \omega, x)$  does not converge for some continuous function  $\varphi : \mathbb{S}^1 \rightarrow \mathbb{R}$ .

**Definition 3.** Let  $f : \Omega \rightarrow \mathcal{C}^r(\mathbb{S}^1, \mathbb{S}^1)$  be a measurable mapping. Let  $X$  be a random variable on  $\Omega$  with values in  $\mathbb{S}^1$ . We call

$$\{X(\omega), f_{\theta^{-1}\omega}(X(\theta^{-1}\omega)), f_{\theta^{-2}\omega}^{(2)}(X(\theta^{-2}\omega)), \dots\}$$

the *random orbit* of  $X$  at  $\omega$ .

We say that  $X$  has *historic behaviour* if there exist real numbers  $\alpha, \beta$  and a continuous function  $\varphi : \mathbb{S}^1 \rightarrow \mathbb{R}$ , which are independent of  $\omega$ , such that

$$(2.1) \quad \liminf_{n \rightarrow \infty} B_n(\varphi; \omega, X(\omega)) < \alpha < \beta < \limsup_{n \rightarrow \infty} B_n(\varphi; \omega, X(\omega))$$

for  $\mathbb{P}$ -almost every  $\omega$ .

Due to the following proposition, Theorem 2 is reduced to constructing *one* measurable mapping  $X$  whose orbit is  $\mathbb{P}$ -almost surely dense and has historic behaviour.

**Proposition 4.** Let  $f : \Omega \rightarrow \mathcal{C}^r(\mathbb{S}^1, \mathbb{S}^1)$  be a measurable mapping. Assume that there exists a measurable mapping  $X : \Omega \rightarrow \mathbb{S}^1$  such that the random orbit of  $X$  is  $\mathbb{P}$ -almost surely dense in  $\mathbb{S}^1$ , and that  $X$  has historic behaviour. Then, for  $\mathbb{P}$ -almost every  $\omega \in \Omega$ , we can find a residual subset  $\mathcal{R}^\omega$  of  $\mathbb{S}^1$  such that for any  $x \in \mathcal{R}^\omega$ , the random orbit issued from  $(\omega, x)$  has historic behaviour.

*Proof.* Let  $\alpha, \beta$  and  $\varphi$  be constants and a continuous function given in (2.1) for  $X$ , which has historic behaviour by hypothesis. We denote by  $\mathcal{R}^\omega$

$$(2.2) \quad \left( \bigcap_{N \geq 1} \bigcup_{n \geq N} \{x \in \mathbb{S}^1 : B_n(\varphi; \omega, x) < \alpha\} \right) \cap \left( \bigcap_{N \geq 1} \bigcup_{n \geq N} \{x \in \mathbb{S}^1 : B_n(\varphi; \omega, x) > \beta\} \right)$$

for each  $\omega \in \Omega$ . For simplicity we write  $s_1^\omega$  and  $s_2^\omega$  for the set in the left big parenthesis and the set in the right big parenthesis in (2.2), respectively, i.e.,  $\mathcal{R}^\omega = s_1^\omega \cap s_2^\omega$ . It is straightforward to see that for each  $\omega \in \Omega$ ,  $\mathcal{R}^\omega$  coincides with the set of  $x \in \mathbb{S}^1$  such that (2.1) holds with  $X(\omega)$  replaced by  $x$ . Consequently, for any  $\omega \in \Omega$  and  $x \in \mathcal{R}^\omega$ , the random orbit issued from  $(\omega, x)$  has historic behaviour. Note also that  $\mathcal{R}^\omega$  is a countable intersection of open sets.

The rest of the proof is devoted to showing that the random orbit of  $X$  at  $\omega$  is included in  $\mathcal{R}^\omega$  for  $\mathbb{P}$ -almost every  $\omega \in \Omega$ . Since the random orbit of  $X$  is  $\mathbb{P}$ -almost surely dense in  $\mathbb{S}^1$  by hypothesis, it leads to that  $\mathcal{R}^\omega$  is dense in  $\mathbb{S}^1$ , and we complete the proof. Set  $\ell \geq 1$ . Due to the observation  $f_\omega^{(i)} \circ f_{\theta^{-\ell}\omega}^{(\ell)} = f_{\theta^{-\ell}\omega}^{(i+\ell)}$  for any  $i \geq 0$ , we

have

$$B_n \left( \varphi; \omega, f_{\theta^{-\ell}\omega}^{(\ell)}(X(\theta^{-\ell}\omega)) \right) = \frac{1}{n} \sum_{j=\ell}^{n+\ell-1} \varphi \circ f_{\theta^{-\ell}\omega}^{(j)}(X(\theta^{-\ell}\omega)).$$

Therefore, for each  $n \geq \ell$

$$\begin{aligned} B_n \left( \varphi; \omega, f_{\theta^{-\ell}\omega}^{(\ell)}(X(\theta^{-\ell}\omega)) \right) - B_n(\varphi; \theta^{-\ell}\omega, X(\theta^{-\ell}\omega)) \\ = \frac{1}{n} \left( \sum_{j=n}^{n+\ell-1} \varphi \circ f_{\theta^{-\ell}\omega}^{(j)}(X(\theta^{-\ell}\omega)) - \sum_{j=0}^{\ell-1} \varphi \circ f_{\theta^{-\ell}\omega}^{(j)}(X(\theta^{-\ell}\omega)) \right). \end{aligned}$$

Hence we have

$$(2.3) \quad \left| B_n \left( \varphi; \omega, f_{\theta^{-\ell}\omega}^{(\ell)}(X(\theta^{-\ell}\omega)) \right) - B_n(\varphi; \theta^{-\ell}\omega, X(\theta^{-\ell}\omega)) \right| \leq \frac{2\ell \|\varphi\|_{\mathcal{C}^0}}{n},$$

which goes to zero as  $n$  goes to infinity for any fixed  $\ell \geq 1$ .

We fix  $\omega \in \Omega$  and  $\ell \geq 1$  for which (2.1) holds with  $\omega$  replaced by  $\theta^{-\ell}\omega$ , i.e.,

$$(2.4) \quad \liminf_{n \rightarrow \infty} B_n(\varphi; \theta^{-\ell}\omega, X(\theta^{-\ell}\omega)) < \alpha < \beta < \limsup_{n \rightarrow \infty} B_n(\varphi; \theta^{-\ell}\omega, X(\theta^{-\ell}\omega)).$$

Let  $\{n_k\}_{k \geq 1}$  be a sequence such that  $B_{n_k}(\varphi; \theta^{-\ell}\omega, X(\theta^{-\ell}\omega))$  converges to the infimum limit in (2.4) as  $k$  goes to infinity. Then, in view of (2.3) we have

$$B_{n_k} \left( \varphi; \omega, f_{\theta^{-\ell}\omega}^{(\ell)}(X(\theta^{-\ell}\omega)) \right) < \alpha \quad \text{if } k \text{ is sufficiently large.}$$

This implies that  $f_{\theta^{-\ell}\omega}^{(\ell)}(X(\theta^{-\ell}\omega))$  is in  $s_1^\omega$ . In a similar manner, we can show that  $f_{\theta^{-\ell}\omega}^{(\ell)}(X(\theta^{-\ell}\omega))$  is in  $s_2^\omega$ . Therefore,  $f_{\theta^{-\ell}\omega}^{(\ell)}(X(\theta^{-\ell}\omega))$  is in  $\mathcal{R}^\omega$ , and the random orbit of  $X$  at  $\omega$  is included in  $\mathcal{R}^\omega$  for  $\mathbb{P}$ -almost every  $\omega \in \Omega$ .  $\square$

**2.1. Shub's Theorem and Markov partition.** In the next subsection, we establish the coding of *graphs* associated with a *random Markov partition* of  $f_\epsilon$ , which is a key ingredient in our proof. For that purpose, we will need the following extension of Shub's Theorem on topological conjugacy between expanding mappings and the folding mapping with the same degree to our random setting. Recall that in view of (1.4), the degree of  $f_\epsilon(\omega, \cdot) : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is  $k$  if  $0 \leq \epsilon < \epsilon_0$  and  $\omega \in \Omega$ .

**Theorem 5.** *Suppose that  $0 \leq \epsilon < \epsilon_0$ . Then we can find a continuous mapping  $h : \Omega \rightarrow \text{Homeo}(\mathbb{S}^1, \mathbb{S}^1)$  (in particular, measurable mapping) which satisfies that*

$$(2.5) \quad h(\theta\omega) \circ E_k = f_\epsilon(\omega) \circ h(\omega), \quad \mathbb{P}\text{-almost surely,}$$

where  $E_k : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is the  $k$ -folding mapping defined by  $E_k(x) = kx \pmod{1}$ .

Furthermore, for  $\mathbb{P}^2$ -almost every  $(\omega, \omega')$  we have

$$(2.6) \quad d_{\mathcal{C}^0}(h(\omega), h(\omega')) \leq \delta_0,$$

where  $\delta_0$  is given in (1.3).

*Proof.* Throughout the proof, we fix  $0 \leq \epsilon < \epsilon_0$  and omit it in notions if there is no confusion. We shall first find a continuous mapping  $p : \Omega \rightarrow \mathbb{S}^1$ , which is invariant under  $f_\epsilon$ , i.e.,  $f_\epsilon(\omega, p(\omega)) = p(\theta\omega)$  for every  $\omega \in \Omega$ . Recall that  $B$  is a closed interval (given above (1.5)) such that  $B \subset f_\epsilon(\omega)(B)$  and  $f_\epsilon(\omega) : B \rightarrow f_\epsilon(\omega)(B)$  is a diffeomorphism for every  $\omega \in \Omega$ . We denote for each  $\omega \in \Omega$  the inverse branch of  $f_\epsilon(\omega)$  restricted on  $B$  by  $F(\omega)$ , i.e.,  $F(\omega) : f_\epsilon(\omega)(B) \rightarrow B$ . Then  $F(\omega) : B \rightarrow B$  is also well defined since  $B \subset f_\epsilon(\omega)(B)$  for every  $\omega \in \Omega$ . Hence, with the notation

of the space of all  $\mathcal{C}^r$  diffeomorphisms on the closed interval  $B$  by  $\text{Diff}^r(B, B)$ , it follows from the continuity of  $f_\epsilon$  that  $F : \Omega \rightarrow \text{Diff}^r(B, B)$  is continuous. Due to (1.2), we have that for each  $0 \leq \epsilon < \epsilon_0$ , we have

$$(2.7) \quad \sup_{\omega} \sup_x \left| \frac{d}{dx} [F(\omega)](x) \right| \leq \lambda^{-1}.$$

Let  $\mathcal{C}^0(\Omega, B)$  be the space of all continuous mappings from  $\Omega$  to the closed interval  $B$ , endowed with the usual  $\mathcal{C}^0$  metric  $d_{\mathcal{C}^0}(\cdot, \cdot)$  defined by  $d_{\mathcal{C}^0}(\phi_1, \phi_2) = \sup_{\omega} |\phi_1(\omega) - \phi_2(\omega)|$  for  $\phi_1, \phi_2 \in \mathcal{C}^0(\Omega, B)$ . For each  $\varphi \in \mathcal{C}^0(\Omega, B)$ , we define a mapping  $\Gamma(\varphi) : \Omega \rightarrow B$  by

$$\Gamma(\varphi)(\omega) = F(\omega)(\varphi(\theta\omega)), \quad \omega \in \Omega.$$

It is straightforward to see that  $(\omega, x) \mapsto F(\omega)(x)$  is a continuous mapping from  $\Omega \times \mathbb{S}^1$  to  $\mathbb{S}^1$  (see the argument below (1.1)). Thus, it follows from the continuity of  $\theta$  and  $\varphi$  that  $\Gamma(\varphi) : \Omega \rightarrow B$  is also a continuous mapping. (The transformation  $\Gamma : \mathcal{C}^0(\Omega, B) \rightarrow \mathcal{C}^0(\Omega, B)$  is called the *graph transformation* induced by  $F$ .) Furthermore, by virtue of (2.7) together with the mean value theorem, we have

$$\begin{aligned} d_{\mathcal{C}^0}(\Gamma(\phi_1), \Gamma(\phi_2)) &\leq \sup_{\omega \in \Omega} |F(\omega)(\phi_1(\theta\omega)) - F(\omega)(\phi_2(\theta\omega))| \\ &\leq \lambda^{-1} \sup_{\omega \in \Omega} |\phi_1(\theta\omega) - \phi_2(\theta\omega)| = \lambda^{-1} d_{\mathcal{C}^0}(\phi_1, \phi_2), \end{aligned}$$

for all  $\phi_1, \phi_2 \in \mathcal{C}^0(\Omega, B)$ , i.e.,  $\Gamma$  is a contraction mapping on the complete metric space  $\mathcal{C}^0(\Omega, B)$ . Therefore, there exists a unique fixed point  $p$  of  $\Gamma$ . By construction,  $p : \Omega \rightarrow B \subset \mathbb{S}^1$  is an  $f_\epsilon$ -invariant continuous mapping.

We next construct a sequence  $\{h_n\}_{n \geq 0}$  of measurable mappings  $h_n : \Omega \rightarrow \text{Homeo}(\mathbb{S}^1, \mathbb{S}^1)$ , which shall converge to the desired mapping  $h$  as  $n$  goes to infinity. Let  $\tilde{f}_\epsilon(\omega) : \mathbb{R} \rightarrow \mathbb{R}$  for  $\omega \in \Omega$  be a lift of the  $k$ -covering  $f_\epsilon(\omega)$  of  $\mathbb{S}^1 = [0, 1]$  such that  $\tilde{f}_\epsilon(\omega)(p(\omega)) = p(\theta\omega)$  and  $\tilde{f}_\epsilon(\omega)(p(\omega) + 1) = p(\theta\omega) + k$ , and that  $\tilde{f}_\epsilon : [p(\omega), p(\omega) + 1] \rightarrow [p(\theta\omega), p(\theta\omega) + k]$  is a monotonically increasing homeomorphism for all  $\omega$ . Now we define points  $a_j^n(\omega)$  in  $[p(\omega), p(\omega) + 1]$  for  $n \geq 0, 0 \leq j \leq k^n - 1$  and  $\omega \in \Omega$ , inductively with respect to  $n$ . In the case  $n = 0$ , let  $a_0^0(\omega) = p(\omega)$  for each  $\omega \in \Omega$ . For given integer  $n \geq 0$ , we assume that  $a_j^n(\omega)$ 's are well defined for each  $\omega \in \Omega$  and  $0 \leq j \leq k^n - 1$ . Then we define  $a_j^{n+1}(\omega)$  for  $0 \leq j \leq k^{n+1} - 1$  of the form  $j = \ell \cdot k^n + j_1$  with integers  $0 \leq \ell \leq k - 1$  and  $0 \leq j_1 \leq k^n - 1$  by

$$(2.8) \quad a_j^{n+1}(\omega) = \left( \tilde{f}_\epsilon(\omega) \right)^{-1} (a_{j_1}^n(\theta\omega) + \ell), \quad \omega \in \Omega.$$

For convenience, we set  $a_{k^n}^n(\omega) = p(\omega) + 1$  for each  $\omega \in \Omega$ .

For the time being, we fix  $n \geq 0$  and  $\omega \in \Omega$ . Then it follows from the monotonicity of  $\tilde{f}_\epsilon(\omega)$  in (2.8) that  $a_0^n(\omega) < a_1^n(\omega) < \cdots < a_{k^n}^n(\omega)$ . Thus we can define a continuous, monotonically increasing, piecewise linear mapping  $\tilde{h}_n(\omega) : [0, 1] \rightarrow [p(\omega), p(\omega) + 1]$  such that

$$(2.9) \quad \tilde{h}_n(\omega) \left( \frac{j}{k^n} \right) = a_j^n(\omega), \quad 0 \leq j \leq k^n,$$

and that  $\tilde{h}_n(\omega) : [j/k^n, (j+1)/k^n] \rightarrow [a_j^n(\omega), a_{j+1}^n(\omega)]$  is an affine mapping of slope  $k^n \cdot (a_{j+1}^n(\omega) - a_j^n(\omega))$ . We finally define  $h_n(\omega) : \mathbb{S}^1 \rightarrow \mathbb{S}^1 = [0, 1]$  by

$$(2.10) \quad h_n(\omega) = \pi_{\mathbb{S}^1} \circ \tilde{h}_n(\omega),$$

where  $\pi_{\mathbb{S}^1}$  is the natural projection from  $\mathbb{R}$  to  $\mathbb{S}^1$ .

Now we change  $n \geq 0$  while  $\omega$  is still fixed. For all positive integers  $n$  and  $m$  with  $n \leq m$ , the supremum norm of  $h_n(\omega)(x) - h_m(\omega)(x)$  (over  $x \in \mathbb{S}^1$ ) is bounded by

$$\max_{0 \leq j \leq k^n - 1} |a_{j+1}^n(\omega) - a_j^n(\omega)|,$$

which converges to zero as  $n$  goes to infinity. Indeed, since  $\pi_{\mathbb{S}^1}([a_j^n(\omega), a_{j+1}^n(\omega)])$  is diffeomorphically mapped onto  $\mathbb{S}^1$  by  $f_\omega^{(n)}$ , the length of each  $[a_j^n(\omega), a_{j+1}^n(\omega)]$  does not exceed  $\lambda^{-n}$  (independently of  $\omega$ ) by virtue of (1.2), so that  $\{a_j^n(\omega) \mid n \geq 0, 0 \leq j \leq k^n\}$  is dense in  $\mathbb{S}^1$ . Hence,  $\{h_n\}_{n \geq 0}$  is a Cauchy sequence with respect to the uniform norm of  $\omega$  and there exists the limit mapping  $h = \lim_{n \rightarrow \infty} h_n$ , which is by construction a continuous mapping from  $\Omega$  to  $\text{Homeo}(\mathbb{S}^1, \mathbb{S}^1)$ .

We see the conjugacy (2.5) between  $f_\epsilon$  and  $E_k$ . Let  $\omega \in \Omega$ ,  $n \geq 0$  and  $0 \leq j \leq k^{n+1} - 1$  of the form  $j = \ell \cdot k^n + j_1$  with some integers  $0 \leq \ell \leq k - 1$  and  $0 \leq j_1 \leq k^n - 1$ . Then on the one hand, in view of (2.8), (2.9) and (2.10), we have

$$f_\epsilon(\omega) \circ h(\omega) \left( \frac{j}{k^{n+1}} \right) = \pi_{\mathbb{S}^1} \circ \tilde{f}_\epsilon(\omega)(a_j^{n+1}(\omega)) = \pi_{\mathbb{S}^1} (a_{j_1}^n(\theta\omega)).$$

On the other hand,

$$h(\theta\omega) \circ E_k \left( \frac{j}{k^{n+1}} \right) = h(\theta\omega) \left( \frac{j_1}{k^n} \right) = \pi_{\mathbb{S}^1} (a_{j_1}^n(\theta\omega)).$$

Since  $\{j/k^n \mid n \geq 0, 0 \leq j \leq k^n - 1\}$  is dense in  $\mathbb{S}^1$ , we get (2.5).

In the end, we prove the estimate (2.6) of  $h$  by showing

$$(2.11) \quad |a_j^n(\omega) - a_j^n(\omega')| \leq \delta_0$$

for all  $n \geq 0$ ,  $0 \leq j \leq k^n - 1$  and  $\omega, \omega' \in \Omega$ . We show (2.11) by induction with respect to  $n \geq 0$ . Due to (1.5), this inequality holds for  $n = 0$ . Suppose that (2.11) is true for given  $n \geq 0$ . Let  $\omega, \omega' \in \Omega$  and  $0 \leq j \leq k^{n+1} - 1$  of the form  $j = \ell \cdot k^n + j_1$  with some integers  $0 \leq \ell \leq k - 1$  and  $0 \leq j_1 \leq k^n - 1$ . Then, it follows from (2.8) together with the triangle inequality that  $|a_j^{n+1}(\omega) - a_j^{n+1}(\omega')|$  is bounded by  $s_1 + s_2$ , where

$$s_1 = \left| \left( \tilde{f}_\epsilon(\omega) \right)^{-1} (a_{j_1}^n(\theta\omega) + \ell) - \left( \tilde{f}_\epsilon(\omega') \right)^{-1} (a_{j_1}^n(\theta\omega) + \ell) \right|,$$

$$s_2 = \left| \left( \tilde{f}_\epsilon(\omega') \right)^{-1} (a_{j_1}^n(\theta\omega) + \ell) - \left( \tilde{f}_\epsilon(\omega') \right)^{-1} (a_{j_1}^n(\theta\omega') + \ell) \right|.$$

To estimate  $s_1$ , we write  $x$  and  $x'$  for the first and second term in the absolute value of  $s_1$ , respectively. Then,  $\tilde{f}_\epsilon(\omega')(x') = \tilde{f}_\epsilon(\omega)(x)$ , and in view of (1.4) we have

$$\left| \tilde{f}_\epsilon(\omega')(x) - \tilde{f}_\epsilon(\omega')(x') \right| = \left| \tilde{f}_\epsilon(\omega')(x) - \tilde{f}_\epsilon(\omega)(x) \right| \leq \eta.$$

Hence, it follows from (1.2) and the mean value theorem that

$$s_1 = |x - x'| \leq \lambda^{-1} \left| \tilde{f}_\epsilon(\omega')(x) - \tilde{f}_\epsilon(\omega')(x') \right| \leq \lambda^{-1} \eta.$$

On the other hand, by virtue of (1.2) and the mean value theorem together with the hypothesis of induction, we get

$$s_2 \leq \lambda^{-1} \left| (a_{j_1}^n(\theta\omega) + \ell) - (a_{j_1}^n(\theta\omega') + \ell) \right| \leq \lambda^{-1} \delta_0.$$



These estimates together with the condition (1.3) on  $\eta$  and  $\delta_0$  implies that  $|a_j^{n+1}(\omega) - a_j^{n+1}(\omega')| \leq \delta_0$ , which completes the proof of the claim (2.11).  $\square$

*Remark.* It is usually desirable to see if any topological condition of base transformation  $\theta$  on the noise space  $(\Omega, \mathbb{P})$  (in our case, the continuity of  $\theta$ ) is removable. This boils down to whether the random conjugacy  $h : \Omega \rightarrow \text{Homeo}(\mathbb{S}^1, \mathbb{S}^1)$  is measurable without the continuity of  $\theta$ . However it is unclear to us whether requiring continuity of  $\theta$  is due to a substantial obstacle or is an artifact of our construction.

Set  $\mathcal{A} = \{0, 1, \dots, k-1\}$ . We need the following proposition for a random Markov partition.

**Definition 6.** Let  $f : \Omega \rightarrow \mathcal{C}^r(\mathbb{S}^1, \mathbb{S}^1)$  be a measurable mapping and denote  $f(\omega)$  by  $f_\omega$ . We say that a finite collection  $\{\mathcal{I}_j^{(\cdot)}\}_{j \in \mathcal{A}}$  of random variables defined on  $\Omega$  with values in  $\mathcal{F}(\mathbb{S}^1)$  is a *Markov partition* of  $f$  if it satisfies the following conditions:

- $\mathcal{I}_j^\omega$  is  $\mathbb{P}$ -almost surely a nonempty left-closed and right-open interval for each  $j \in \mathcal{A}$ ,
- $\bigsqcup_{j \in \mathcal{A}} \mathcal{I}_j^\omega = \mathbb{S}^1$  for  $\mathbb{P}$ -almost every  $\omega \in \Omega$  under the identification  $\mathbb{S}^1 \cong [0, 1)$ ,
- $f_\omega(\mathcal{I}_j^\omega) = \mathbb{S}^1$  and  $f_\omega : \mathcal{I}_j^\omega \rightarrow \mathbb{S}^1$  is a  $\mathcal{C}^r$  diffeomorphism for each  $j \in \mathcal{A}$  and  $\mathbb{P}$ -almost every  $\omega \in \Omega$ ,
- $(f_\omega)^{-1}(\mathcal{I}_i^{\theta\omega}) \cap \mathcal{I}_j^\omega$  is a nonempty left-closed and right-open interval for each  $i, j \in \mathcal{A}$  and  $\mathbb{P}$ -almost every  $\omega \in \Omega$ .

**Proposition 7.** Suppose that  $0 \leq \epsilon < \epsilon_0$ . Then there is a Markov partition  $\{\mathcal{I}_j^{(\cdot)}\}_{j=0}^{k-1}$  of  $f_\epsilon$  such that for each  $0 \leq j \leq k-1$ , we can find a nonempty open interval  $J'$  such that  $\mathcal{I}_j^\omega$  does not intersect  $J'$  for  $\mathbb{P}$ -almost every  $\omega \in \Omega$ .

*Proof.* Let  $I_j = [j/k, (j+1)/k)$  for  $0 \leq j \leq k-1$ . Then  $\{I_j\}_{j=0}^{k-1}$  is a Markov partition of the  $k$ -folding map  $E_k$ . I.e.,

- $I_j$  is a left-closed and right-open interval for each  $j \in \mathcal{A}$ ,
- $\bigsqcup_{j \in \mathcal{A}} I_j = \mathbb{S}^1$ ,
- $E_k(I_j) = \mathbb{S}^1$  and  $E_k : I_j \rightarrow \mathbb{S}^1$  is a  $\mathcal{C}^r$  diffeomorphism,
- $(E_k)^{-1}(I_i) \cap I_j$  is a left-closed and right-open interval for each  $i, j \in \mathcal{A}$ .

We fix  $0 \leq \epsilon < \epsilon_0$ , and let  $h : \Omega \rightarrow \text{Homeo}(\mathbb{S}^1, \mathbb{S}^1)$  be the measurable mapping satisfying (2.5) in Theorem 5. Set  $\mathcal{I}_j^\omega = h(\omega)(I_j)$  for each  $\omega \in \Omega$  and  $0 \leq j \leq k-1$ .

Then due to Theorem 5, it is straightforward to see that  $\{\mathcal{I}_j^{(\cdot)}\}_{j=0}^{k-1}$  is a Markov partition of  $f_\epsilon$  such that

$$\text{ess sup}_{(\omega, \omega') \in \Omega^2} d_H(\mathcal{I}_j^\omega, \mathcal{I}_j^{\omega'}) \leq \frac{1}{k} + 2\delta_0 < 1$$

for each  $0 \leq j \leq k-1$ , where the last inequality follows from the condition of  $\delta_0$  given above (1.3). This immediately implies the conclusion.  $\square$

**2.2. Coding of graphs.** Throughout this subsection, we fix  $0 \leq \epsilon < \epsilon_0$ , and let  $\{\mathcal{I}_j^{(\cdot)}\}_{j=0}^{k-1}$  be the Markov partition of  $f_\epsilon$  in Theorem 7. For a word  $s = (s_0, s_1, s_2, \dots, s_{n-1}) \in \mathcal{A}^n$  and  $\omega \in \Omega$ , let  $\mathcal{I}_s^\omega$  be a subset of  $\mathbb{S}^1$  defined by

$$(2.12) \quad \mathcal{I}_s^\omega = \bigcap_{j=0}^{n-1} \left( f_\omega^{(j)} \right)^{-1} \left( \mathcal{I}_{s_j}^{\theta^j \omega} \right).$$



**Lemma 8.** *For each  $n \geq 1$  and  $s \in \mathcal{A}^n$ ,  $\mathcal{I}_s^\omega$  is a nonempty left-closed and right-open interval of  $\mathbb{S}^1$  for  $\mathbb{P}$ -almost every  $\omega \in \Omega$ . Moreover, the mapping  $\omega \mapsto \mathcal{I}_s^\omega$  from  $\Omega$  to  $\mathcal{F}(\mathbb{S}^1)$  is measurable.*

*Proof.* We first prove that  $\mathcal{I}_s^\omega$  is a nonempty left-closed and right-open interval by induction. It is true for  $n = 1$  due to Proposition 7. Assume that the claim is true for a positive integer  $n$ , i.e., the set  $\mathcal{I}^\omega$  is a nonempty left-closed and right-open interval for  $\mathbb{P}$ -almost every  $\omega \in \Omega$  and  $s \in \mathcal{A}^n$ . Then, for  $\mathbb{P}$ -almost every  $\omega \in \Omega$  and  $s = (s_0, s_1, \dots, s_n) \in \mathcal{A}^{n+1}$ , using again  $(f_\omega^{(j)})^{-1} = (f_\omega)^{-1} \circ (f_{\theta\omega}^{(j-1)})^{-1}$  for all  $j \geq 1$ , we have

$$\begin{aligned}
 (2.13) \quad \mathcal{I}_s^\omega &= \mathcal{I}_{s_0}^\omega \cap \left( (f_\omega)^{-1} \left( \bigcap_{j=1}^n (f_{\theta\omega}^{(j-1)})^{-1} (\mathcal{I}_{s_j}^{\theta^j \omega}) \right) \right) \\
 &= \mathcal{I}_{s_0}^\omega \cap \left( (f_\omega)^{-1} \left( \bigcap_{j=0}^{n-1} (f_{\theta\omega}^{(j)})^{-1} (\mathcal{I}_{s_{j+1}}^{\theta^{j+1} \omega}) \right) \right) \\
 &= \mathcal{I}_{s_0}^\omega \cap \left( (f_\omega)^{-1} (\mathcal{I}_{\sigma(s)}^{\theta\omega}) \right),
 \end{aligned}$$

where  $\sigma : \mathcal{A}^{n+1} \rightarrow \mathcal{A}^n$  is the one-sided shift defined by  $\sigma(s) = (s_1, s_2, \dots, s_n)$  for  $s = (s_0, s_1, \dots, s_n) \in \mathcal{A}^{n+1}$ . Note that  $f_\omega : \mathcal{I}_{s_0}^\omega \rightarrow \mathbb{S}^1$  is a diffeomorphism, and  $\mathcal{I}_{\sigma(s)}^{\theta\omega}$  is a nonempty left-closed and right-open interval (for  $\mathbb{P}$ -almost every  $\omega \in \Omega$ ) by the hypothesis of induction. Hence,  $\mathcal{I}_s^\omega$  is also a nonempty left-closed and right-open interval, and we complete the proof of the claim.

We next prove the measurability of  $\mathcal{I}_s^\omega$ . Note that  $\omega \mapsto (f_\omega^{(j)})^{-1}(\mathcal{I}_{s_j}^{\theta^j \omega})$  is the composition of a mapping  $\alpha : \Omega \times \Omega \rightarrow \mathcal{F}(\mathbb{S}^1)$  given by

$$\alpha(\omega, \omega') = (f_\omega^{(j)})^{-1}(\mathcal{I}_{s_j}^{\theta^j \omega'}), \quad (\omega, \omega') \in \Omega \times \Omega$$

and a measurable mapping  $\Lambda : \Omega \rightarrow \Omega \times \Omega$  given by  $\Lambda(\omega) = (\omega, \omega)$  for  $\omega \in \Omega$ . Thus, in view of (2.12), it suffices to show the measurability of  $\alpha : \Omega \times \Omega \rightarrow \mathcal{F}(\mathbb{S}^1)$ .

We fix  $n \geq 1$ ,  $s \in \mathcal{A}^n$  and  $0 \leq j \leq n$ . It follows from the continuity of  $\Omega \times \mathbb{S}^1 \ni (\omega, x) \mapsto f_\omega(\omega, x)$  and  $\theta$  that for each  $I \in \mathcal{F}$ , the mapping  $\omega \mapsto (f_\omega^{(j)})^{-1}(I)$  is a continuous mapping from  $\Omega$  to  $\mathcal{F}(\mathbb{S}^1)$ . Therefore  $\omega \mapsto \alpha(\omega, \omega')$  is a continuous mapping from  $\Omega$  to  $\mathcal{F}(\mathbb{S}^1)$  for each  $\omega' \in \Omega$ . On the other hand, the mapping  $\Omega \ni \omega' \mapsto \alpha(\omega, \omega')$  is measurable for each  $\omega \in \Omega$  since  $(f_\omega^{(j)})^{-1} : \mathcal{F}(\mathbb{S}^1) \rightarrow \mathcal{F}(\mathbb{S}^1)$  is continuous for each  $\omega \in \Omega$  and  $\mathcal{I}_j^{(\cdot)}$  is measurable for each  $0 \leq j \leq k-1$ . Hence, by [6, Lemma 3.14],  $\alpha : \Omega \times \Omega \rightarrow \mathcal{F}(\mathbb{S}^1)$  is measurable.  $\square$

For a word  $s = (s_0, s_1, \dots) \in \mathcal{A}^{\mathbb{N}_0}$  of infinite length, we define the interval  $\mathcal{I}_s^\omega$  by  $\mathcal{I}_s^\omega = \bigcap_{n \geq 0} \mathcal{I}_{[s]_n}^\omega$ , where  $[s]_n = (s_0, s_1, \dots, s_{n-1})$ . Then, since the inverse branches of  $f_\omega$  are contractions due to (1.2), it follows from Lemma 8 that  $\mathcal{I}_s^\omega$  is a point set. We denote the point by  $X_s(\omega)$ . Then, by the measurability in Lemma 8, the mapping  $\omega \mapsto \{X_s(\omega)\}$  from  $\Omega$  to  $\mathcal{F}(\mathbb{S}^1)$  is measurable, so is the mapping  $X_s : \Omega \rightarrow \mathbb{S}^1$ .

The following lemmas on the graphs of  $X_s$ 's are simple but substantial in the proof of Theorem 2. Let  $\sigma : \mathcal{A}^{\mathbb{N}_0} \rightarrow \mathcal{A}^{\mathbb{N}_0}$  be the one-sided shift given by  $\sigma(\omega) = (\omega_1, \omega_2, \dots)$  for  $\omega = (\omega_0, \omega_1, \dots) \in \mathcal{A}^{\mathbb{N}_0}$ .

**Lemma 9.** *For any  $s \in \mathcal{A}^{\mathbb{N}_0}$ , we  $\mathbb{P}$ -almost surely have*

$$f_\omega(X_s(\omega)) = X_{\sigma(s)}(\theta\omega),$$

*and we can find a nonempty open interval  $J'$  such that  $X_s(\omega)$  does not intersect  $J'$  for  $\mathbb{P}$ -almost every  $\omega \in \Omega$ .*

*Proof.* From (2.13), it follows that

$$\begin{aligned} f_\omega(\mathcal{I}_{[s]_n}^\omega) &= f_\omega(\mathcal{I}_{s_0}^\omega) \cap \mathcal{I}_{\sigma([s]_n)}^{\theta\omega} \\ &= \mathbb{S}^1 \cap \mathcal{I}_{\sigma([s]_n)}^{\theta\omega} = \mathcal{I}_{\sigma([s]_n)}^{\theta\omega}. \end{aligned}$$

Thus, by the virtue of the continuity of  $f_\omega : \mathcal{F}(\mathbb{S}^1) \rightarrow \mathcal{F}(\mathbb{S}^1)$ , we get  $f_\omega(\mathcal{I}_s^\omega) = \mathcal{I}_{\sigma(s)}^{\theta\omega}$  for  $\mathbb{P}$ -almost every  $\omega \in \Omega$ . Together with the construction of  $X_s$  and Proposition 7, this implies the conclusion.  $\square$

**Lemma 10.** *Let  $x$  be a point in  $\mathbb{S}^1$  and  $s = s(x) = (s_0, s_1, \dots) \in \mathcal{A}^{\mathbb{N}_0}$  the coding of  $x$  by  $E_k$ , i.e.,  $E_k^j(x) \in I_{s_j}$  for each  $j \geq 0$ . Then,  $h(\omega)(x) = X_s(\omega)$  for  $\mathbb{P}$ -almost every  $\omega \in \Omega$  and  $x \in \mathbb{S}^1$ , where  $h$  is given in Theorem 5.*

*Proof.* It follows from (2.5) and (2.12) that for each  $n \geq 1$ , we have

$$\begin{aligned} \mathcal{I}_{[s]_n}^\omega &= \bigcap_{j=0}^{n-1} \left( f_\omega^{(j)} \right)^{-1} \circ h(\theta^j \omega) (I_{s_j}) \\ &= \bigcap_{j=0}^{n-1} \left( h(\omega) \circ E_k^{-j}(I_{s_j}) \right). \end{aligned}$$

Since  $h(\omega) : \mathbb{S}^1 \rightarrow \mathbb{S}^1$  is a homeomorphism, this implies

$$\mathcal{I}_s^\omega = h(\omega) \left( \bigcap_{j=0}^{\infty} E_k^{-j}(I_{s_j}) \right).$$

Recalling  $\mathcal{I}_s^\omega = \{X_s(\omega)\}$ , we immediately get the conclusion.  $\square$

**2.3. Invariant measures.** As a final preparation before turning to the proof of Theorem 2, we may need to find a sequence  $s''$  such that the orbit of  $X_{s''}$  is  $\mathbb{P}$ -almost surely dense, and the empirical measure along the orbit issued from  $(\omega, X_{s''}(\omega))$   $\mathbb{P}$ -almost surely converges to the unique absolutely ergodic invariant probability measure of  $f_\epsilon$ . For this purpose, we borrow language from L. Arnold [2].

Let  $f : \Omega \rightarrow \mathcal{C}^r(\mathbb{S}^1, \mathbb{S}^1)$  be a measurable mapping. Let  $\mathcal{B}(\mathbb{S}^1)$  be the Borel  $\sigma$ -field of  $\mathbb{S}^1$ . Recall that a measure  $\mu$  on  $\Omega \times \mathbb{S}^1$  is called  $f$ -invariant when  $\mu$  is invariant with respect to the skew product mapping  $\Theta(\omega, x) = (\theta\omega, f(\omega, x))$  and the marginal  $\pi_\Omega \mu$  of  $\mu$  coincides with  $\mathbb{P}$ , where  $\pi_\Omega : \Omega \times \mathbb{S}^1 \rightarrow \Omega$  is given by  $\pi_\Omega(\omega, x) = \omega$ . It is known that when  $\mu$  is an  $f$ -invariant probability measure, there is a unique function  $\mu_\omega(\cdot) : \Omega \times \mathcal{B}(\mathbb{S}^1) \rightarrow [0, 1]$ ,  $(\omega, B) \mapsto \mu_\omega(B)$ , such that

- (i)  $\omega \mapsto \mu_\omega(B)$  is measurable for each  $B \in \mathcal{B}(\mathbb{S}^1)$ ,
- (ii)  $\mu_\omega$  is  $\mathbb{P}$ -almost surely a probability measure on  $\mathbb{S}^1$ ,
- (iii)  $\int u d\mu = \int u d\mu_\omega d\mathbb{P}$  for each  $u \in L^1(\mu)$ .

Moreover, since we assume that  $\theta$  is measurably invertible, the pushforward  $f(\omega)_* \mu_\omega$  of  $\mu_\omega$  by  $f(\omega)$   $\mathbb{P}$ -almost surely coincides with  $\mu_{\theta\omega}$ , see Arnold [2, Chapter 1]. We call the function  $\mu_\omega(\cdot)$  the disintegration of  $\mu$  (with respect to  $\mathbb{P}$ ).

The following theorem for the unique absolutely continuous ergodic invariant measure (abbreviated by *aceip*) is an immediate consequence of the established result by Baladi et al. [3].

**Theorem 11.** *We can find a positive number  $\epsilon_0$  such that for each  $0 \leq \epsilon < \epsilon_0$ , there exists an  $f_\epsilon$ -invariant probability measure  $\mu^\epsilon$  on  $\Omega \times \mathbb{S}^1$  such that if  $\mu^\epsilon(\cdot)$  is the disintegration of  $\mu^\epsilon$  then  $\mu_\omega^\epsilon$  is  $\mathbb{P}$ -almost surely equivalent to normalized Lebesgue measure  $m$  on  $\mathbb{S}^1$ . Moreover, for  $(\mathbb{P} \times m)$ -almost every  $(\omega, x)$ , we have*

$$(2.14) \quad \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f_\epsilon^{(j)}(\omega, x) = \int \varphi d\mu^\epsilon$$

for any continuous mapping  $\varphi : \mathbb{S}^1 \rightarrow \mathbb{R}$ .

*Proof.* Theorem A in [3] guarantees the existence of an  $f_\epsilon$ -invariant probability measure  $\mu^\epsilon$  on  $\Omega \times \mathbb{S}^1$  such that  $\mu_\omega^\epsilon$  is  $\mathbb{P}$ -almost surely equivalent to normalized Lebesgue measure  $m$ , if  $0 \leq \epsilon < \epsilon_0$  is sufficiently small. Hence, it suffices to show that (2.14)  $\mu$ -almost surely holds. On the other hand, it follows from [3, Theorem B] that we can find constants  $0 < \tau < 1$  and  $C > 0$  such that if  $0 \leq \epsilon < \epsilon_0$  is sufficiently small, then for each  $\phi_1 \in L^1(\mathbb{S}^1)$  and  $\phi_2 \in \mathcal{C}^{r-1}(\mathbb{S}^1)$ , we  $\mathbb{P}$ -almost surely have

$$(2.15) \quad |\text{Cor}_{\phi_1, \phi_2}(\omega; n)| \leq C\tau^n \|\phi_1\|_{L^1} \|\phi_2\|_{\mathcal{C}^{r-1}}, \quad n \geq 1,$$

where  $L^1(\mathbb{S}^1)$  is the usual  $L^1$  space on  $\mathbb{S}^1$  endowed with the  $L^1$  norm  $\|\cdot\|_{L^1}$  and  $\mathcal{C}^{r-1}(\mathbb{S}^1)$  is the space of all functions of class  $\mathcal{C}^{r-1}$  endowed with  $\|\cdot\|_{\mathcal{C}^{r-1}}$ , and  $\text{Cor}_{\phi_1, \phi_2}(\omega; \cdot)$  is the quenched correlation function of  $\phi_1, \phi_2$  at  $\omega$  defined by

$$\text{Cor}_{\phi_1, \phi_2}(\omega; n) = \int \phi_1 \circ f_\epsilon^{(n)}(\omega) \phi_2 d\mu_\omega^\epsilon - \int \phi_1 d\mu_{\theta^n \omega}^\epsilon \int \phi_2 d\mu_\omega^\epsilon, \quad n \geq 1.$$

It is straightforward to see

$$(2.16) \quad \frac{1}{n} \sum_{j=0}^{n-1} \text{Cor}_{\phi_1, \phi_2}(\omega; j) = \int \left( \frac{1}{n} \sum_{j=0}^{n-1} \phi_1 \circ f_\epsilon^{(j)}(\omega) - \frac{1}{n} \sum_{j=0}^{n-1} \int \phi_1 d\mu_{\theta^j \omega}^\epsilon \right) \phi_2 d\mu_\omega^\epsilon.$$

Hence it follows from (2.15) that the absolute value of (2.16) is  $\mathbb{P}$ -almost surely bounded by  $Cn^{-1} \sum_{j=0}^{n-1} \tau^j \|\phi_1\|_{L^1} \|\phi_2\|_{\mathcal{C}^{r-1}}$ , which converges to zero as  $n$  goes to infinity. Therefore, since  $\phi_2$  is an arbitrary function of class  $\mathcal{C}^{r-1}$  and  $\mu_\omega$  is equivalent to  $m$ , the dominated convergence theorem leads to that for  $(\mathbb{P} \times m)$ -almost every  $(\omega, x) \in \Omega \times \mathbb{S}^1$ ,

$$\lim_{n \rightarrow \infty} \left( \frac{1}{n} \sum_{j=0}^{n-1} \phi_1 \circ f_\epsilon^{(j)}(\omega)(x) - \frac{1}{n} \sum_{j=0}^{n-1} \int \phi_1 d\mu_{\theta^j \omega}^\epsilon \right) = 0.$$

By Birkoff's ergodic theorem for the ergodic transformation  $\theta : (\Omega, \mathbb{P}) \rightarrow (\Omega, \mathbb{P})$ ,  $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \int \phi_1 d\mu_{\theta^j \omega}^\epsilon$  coincides with  $\int (\int \phi_1 d\mu_\omega^\epsilon) d\mathbb{P} = \int \phi_1 d\mu^\epsilon$  for  $\mathbb{P}$ -almost every  $\omega \in \Omega$ , and thus, the conclusion follows from that  $\phi_1 \in L^1(\mathbb{S}^1)$  is arbitrary and  $\mathcal{C}^0(\mathbb{S}^1)$  is included in  $L^1(\mathbb{S}^1)$ .  $\square$

Note that Theorem 11 may *not* imply in general the existence of a sequence  $s'' \in \mathcal{A}^{\mathbb{N}_0}$  such that the point  $(\omega, X_{s''}(\omega))$   $\mathbb{P}$ -almost surely satisfies (2.14) with  $x$

replaced by  $X_{s''}(\omega)$ . Hence, we need Theorem 12 which ensures the existence of a sequence  $s''$  such that the orbit of  $X_{s''}$  is  $\mathbb{P}$ -almost surely dense.

We prepare language for Theorem 12. Fix  $0 \leq \epsilon < \epsilon_0$  and let  $h = h_\epsilon$  be the mapping given in Theorem 5. Then,  $(\omega, x) \mapsto h(\omega)(x)$  is a continuous mapping from  $\Omega \times \mathbb{S}^1$  to  $\mathbb{S}^1$  due to the continuity of  $h : \Omega \rightarrow \text{Homeo}(\mathbb{S}^1, \mathbb{S}^1)$ . Therefore for each continuous function  $\varphi : \mathbb{S}^1 \rightarrow \mathbb{R}$ , the function  $\Phi_h : \Omega \times \mathbb{S}^1 \rightarrow \mathbb{R}$  given by  $\Phi_h(\omega, x) = \varphi(h(\omega)(x))$  for  $(\omega, x) \in \Omega \times \mathbb{S}^1$  is continuous, in particular measurable. Moreover, it is straightforward to see that  $\|\Phi_h\|_{L^1_{m \times \mathbb{P}}} \leq \|\varphi\|_{\mathcal{C}^0}$ , i.e.,  $\Phi_h$  is integrable function with respect to the product measure  $m \times \mathbb{P}$ . Thus it follows from Fubini's theorem that  $\omega \mapsto \int \Phi_h(\omega, \cdot) dm$  is measurable. Furthermore, since  $\varphi$  is an arbitrary continuous function, it follows from Riesz representation theorem that we can find a probability measure  $(h(\omega))_* m$  on  $\mathbb{S}^1$  such that  $\int \Phi_h(\omega, \cdot) dm = \int \varphi d[(h(\omega))_* m]$  for each  $\omega \in \Omega$ . (This probability measure is called the pushforward of  $m$  by  $h(\omega)$ .)

**Theorem 12.** *For any continuous function  $\varphi : \mathbb{S}^1 \rightarrow \mathbb{R}$ , there exist full measure sets  $\Gamma$  and  $A$  of  $(\Omega, \mathbb{P})$  and  $(\mathbb{S}^1, m)$ , respectively, such that for each  $(\omega, x) \in \Gamma \times A$ , if we set  $s = s(x) \in \mathcal{A}^{\mathbb{N}_0}$  as the coding of  $x$  by  $E_k$  given in Lemma 10, then we have*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f_\omega^{(j)}(X_s(\omega)) = \int \left( \int \varphi d[(h(\cdot))_* m] \right) d\mathbb{P}.$$

*Proof.* For given  $x \in \mathbb{S}^1$ , let  $s \in \mathcal{A}^{\mathbb{N}_0}$  be the coding of  $x$  by  $E_k$ . By Theorem 5 and Lemma 10, we have

$$\varphi \circ f_\omega^{(j)}(X_s(\omega)) = \varphi(f_\omega^{(j)} \circ h(\omega)(x)) = \varphi(h(\theta^j \omega) \circ E_k^j(x))$$

for all  $j \geq 0$  and  $\mathbb{P}$ -almost every  $\omega \in \Omega$ . Thus, if we define a measurable mapping  $\Theta_{E_k} : \Omega \times \mathbb{S}^1 \rightarrow \Omega \times \mathbb{S}^1$  of the direct-product form

$$\Theta_{E_k}(\omega, x) = (\theta\omega, E_k(x)), \quad (\omega, x) \in \Omega \times \mathbb{S}^1$$

then

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \varphi \circ f_\omega^{(j)}(X_s(\omega)) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{j=0}^{n-1} \Phi_h \circ \Theta_{E_k}^j(\omega, x),$$

where  $\Phi_h(\omega, x) = \varphi(h(\omega)(x))$  for  $(\omega, x) \in \Omega \times \mathbb{S}^1$ . This  $(\mathbb{P} \times m)$ -almost surely coincides with  $\int \Phi_h dm d\mathbb{P}$  by virtue of the ergodicity of the direct-product mapping  $\Theta_{E_k}$  with respect to the product measure  $\mathbb{P} \times m$ , and the conclusion immediately follows.  $\square$

*Remark.* Theorem 12 together with Theorem 11 gives information on the geometry of the set of points for which (2.14) holds. That is, it is easy to see that a full measure set  $\{(\omega, h(\omega)(x)) \mid (\omega, x) \in \Gamma \times A\}$  is included in the set of points satisfying (2.14) (recall Lemma 10). Furthermore, these results lead to a formula for the disintegration of the unique aceip  $\mu^\epsilon$ : we have

$$\mu_\omega^\epsilon = (h(\omega))_* m, \quad \mathbb{P}\text{-almost surely.}$$

**2.4. The end of the proof.** We will construct a sequence  $s \in \mathcal{A}^{\mathbb{N}_0}$  such that  $X_s$  satisfies the hypotheses in Proposition 4. As in the treatment undertaken by Takens [11, Section 4], this sequence will be an appropriate combination of a periodic sequence and a sequence generating the unique aceip.

Let  $s' \in \mathcal{A}^{\mathbb{N}_0}$  be a periodic sequence. For simplicity, we set  $s' = (00\dots)$ . By virtue of Proposition 7 and Lemma 9, there exist nonempty open intervals  $J \subset$

$J' \subset \mathbb{S}^1$  both of which do not intersect  $\mathcal{I}_0^\omega$  (in particular,  $X_{s'}(\omega) = X_{(00\dots)}(\omega)$ ) for  $\mathbb{P}$ -almost every  $\omega$ , where the inclusion  $J \subset J'$  is strict. Let  $\varphi_0 : \mathbb{S}^1 \rightarrow \mathbb{R}$  be a  $\mathcal{C}^1$  function such that

- the support of  $\varphi_0$  is included in  $J'$ ,
- $\varphi_0(x) = 1$  for all  $x \in J$ ,
- $0 \leq \varphi(x) \leq 1$  for all  $x \in \mathbb{S}^1$ .

Let  $j \geq 0$  and  $m \geq 0$  be given integers. Then there exists a positive integer  $n_1(j, m)$  such that we  $\mathbb{P}$ -almost surely have

$$(2.17) \quad \sup_{t \in \mathcal{A}^m} \sup_{\tilde{t} \in \mathcal{A}^{\mathbb{N}_0}} \left| B_{n+m}(\varphi_0; \omega, X_{t[s']_n \tilde{t}}(\omega)) \right| \leq \frac{1}{2^{j-1}}$$

for any  $n \geq n_1(j, m)$ , where  $t[s']_n \tilde{t} = (t_0, t_1, \dots, \tilde{s}_{m-1}, s'_0 s'_1, \dots, s'_{n-1}, t_0, t_1, \dots)$  for  $t = (t_0, t_1, \dots, t_{m-1}) \in \mathcal{A}^m$  and  $\tilde{t} = (\tilde{t}_0, \tilde{t}_1, \dots) \in \mathcal{A}^{\mathbb{N}_0}$ . ( $\mathcal{A}^0$  is interpreted as the empty set.) Indeed note that

$$(2.18) \quad B_{n+m}(\varphi_0; \omega, x) = \frac{m}{n+m} B_m(\varphi_0; \omega, x) + \frac{n}{n+m} B_n(\varphi_0; \theta^m \omega, f_\omega^{(m)}(x))$$

for any  $x \in \mathbb{S}^1$ . Moreover, if  $x \in \mathcal{I}_{t[s']_n}^\omega$  with some  $t \in \mathcal{A}^m$  then  $f_\omega^{(m+\ell)}(x) \notin J'$  for all  $0 \leq \ell \leq n-1$  and for  $\mathbb{P}$ -almost every  $\omega \in \Omega$  (since  $f_\omega^{(m+\ell)}(x) \in \mathcal{I}_0^{\theta^{m+\ell} \omega}$  and  $\mathcal{I}_0^{\theta^{m+\ell} \omega} \cap J' = \emptyset$ ). Thus we get  $B_n(\varphi_0; \theta^m \omega, f_\omega^{(m)}(x)) = 0$ , and the claim immediately follows.

On the other hand, it follows from Theorem 11 and 12 together with the remark below these theorems that we can find a sequence  $s'' \in \mathcal{A}^{\mathbb{N}_0}$  such that  $B_n(\varphi_0; \omega, X_{s''}(\omega))$  converges to  $\int \varphi_0 d\mu^\epsilon$  as  $n$  goes to infinity for  $\mathbb{P}$ -almost every  $\omega \in \Omega$ . Moreover, in view of (2.18), for any given sequence  $t \in \mathcal{A}^m$  of finite length  $m$ ,  $B_{[\frac{n}{2}]+m}(\varphi_0; \omega, X_{ts''}(\omega))$  also  $\mathbb{P}$ -almost surely converges to  $\int \varphi_0 d\mu^\epsilon$  as  $n$  goes to infinity, where  $[\frac{n}{2}]$  is the integer part of  $\frac{n}{2}$ . This observation leads to that if we define  $\Gamma_n(j, m)$  for integers  $n \geq 1$ ,  $j \geq 0$  and  $m \geq 0$  by

$$(2.19) \quad \Gamma_n(j, m) = \bigcap_{t \in \mathcal{A}^m} \left\{ \omega \in \Omega : \left| B_{[\frac{n}{2}]+m}(\varphi_0; \omega, X_{ts''}(\omega)) - \int \varphi_0 d\mu^\epsilon \right| \leq \frac{1}{2^j} \right\},$$

then for any integers  $j \geq 0$  and  $m \geq 0$ , we can find a positive integer  $n_2(j, m)$  such that

$$(2.20) \quad \mathbb{P}(\Gamma_n(j, m)) \geq 1 - \frac{1}{2^{j-1}}$$

for any  $n \geq n_2(j, m)$ .

Furthermore, for all integers  $j \geq 0$  and  $m \geq 0$ , there exists a positive integer  $\tilde{n}_3(j, m)$  such that we  $\mathbb{P}$ -almost surely have

$$\sup_{t \in \mathcal{A}^m} \sup_{\tilde{t} \in \mathcal{A}^{\mathbb{N}_0}} \left| B_{[\frac{n}{2}]+m}(\varphi_0; \omega, X_{ts''}(\omega)) - B_{[\frac{n}{2}]+m}(\varphi_0; \omega, X_{t[s']_n \tilde{t}}(\omega)) \right| \leq \frac{1}{2^j}$$

for all  $n \geq \tilde{n}_3(j, m)$ . Indeed, the length of  $\mathcal{I}_{[s']_n}^\omega$  is  $\mathbb{P}$ -almost surely less than  $\lambda^{-n-j} \leq \lambda^{-\frac{n}{2}+1}$  for all  $0 \leq j \leq [\frac{n}{2}]$ . Thus, it follows from the mean value theorem that

$$\left| B_{[\frac{n}{2}]}(\varphi_0; \theta^m \omega, f_\omega^{(m)}(x)) - B_{[\frac{n}{2}]}(\varphi_0; \theta^m \omega, f_\omega^{(m)}(y)) \right| \leq \lambda^{-\frac{n}{2}+1} \|\varphi\|_{\mathcal{C}^1}$$

for any  $x \in \mathcal{I}_{ts''}^\omega$  and  $y \in \mathcal{I}_{t[s'']_n \tilde{t}}^\omega$ . So the claim immediately follows from (2.18). Combining this estimate with the inequality in (2.19), we get that for any  $m \geq 0$  and  $j \geq 0$ , there exists a positive integer  $n_3(j, m)$  such that

$$(2.21) \quad \sup_{t \in \mathcal{A}^m} \sup_{\tilde{t} \in \mathcal{A}^{N_0}} \left| B_{[\frac{m}{2}] + m}(\varphi_0; \omega, X_{t[s'']_n \tilde{t}}(\omega)) - \int \varphi_0 d\mu^\epsilon \right| \leq \frac{1}{2^{j-1}}, \quad \omega \in \Gamma_n(j, m)$$

for any  $n \geq n_3(j, m)$ .

Finally, note that  $X_{s''}$  is  $\mathbb{P}$ -almost surely dense in  $\mathbb{S}^1$  since the empirical measure

$$\frac{1}{n} \sum_{j=0}^{n-1} \delta_{f_\omega^{(j)}(X_{s''}(\omega))}$$

converges to the smooth measure  $\bar{\mu}^\epsilon$ , defined by

$$\bar{\mu}^\epsilon(A) = \int \mu_\omega^\epsilon(A) d\mathbb{P} \quad \text{for } A \in \mathcal{B}(\mathbb{S}^1),$$

due to Theorem 11 and 12.

Now we construct a sequence  $s$  determining the orbit with historic behaviour, which is of the form

$$s = (s_0, \dots, s_{N_1-1}, s_{N_1}, \dots, s_{N_2-1}, s_{N_2}, \dots),$$

where each sequence  $(s_{N_{j-1}}, \dots, s_{N_j-1})$  consists of an initial segment of the sequence  $s'$  if  $j$  is even, and  $s''$  if  $j$  is odd. The length  $\{\tilde{N}_j\}_{j \geq 0}$  of each segment (i.e.,  $\tilde{N}_j = N_j - N_{j-1}$ ) is chosen inductively with respect to  $j$ : for given  $\tilde{N}_1, \tilde{N}_2, \dots, \tilde{N}_{j-1}$ , let  $\tilde{N}_j$  be a positive integer larger than the maximum of  $n_1(j, N_{j-1})$ ,  $n_2(j, N_{j-1})$  and  $n_3(j, N_{j-1})$ , where  $N_{j-1} = \tilde{N}_1 + \tilde{N}_2 + \dots + \tilde{N}_{j-1}$ , with  $N_0 = 0$ . Set

$$\Gamma_0 = \bigcup_{j \geq 1} \Gamma_{\tilde{N}_j}(j, [s]_{N_{j-1}}).$$

Then, it follows from (2.20) that  $\mathbb{P}(\Gamma_0) = 1$ , and by virtue of (2.21), we have

$$\limsup_n B_n(\varphi_0; \omega, X_s(\omega)) \geq \int \varphi_0 d\mu > 0, \quad \omega \in \Gamma_0.$$

Moreover, it follows from (2.17) that

$$\liminf_n B_n(\varphi_0; \omega, X_s(\omega)) \leq 0, \quad \omega \in \Gamma_0.$$

That is,  $X_s$  has historic behaviour. Now the orbit of  $X_s$  is  $\mathbb{P}$ -almost surely dense since the orbit of  $X_{s''}$  is  $\mathbb{P}$ -almost surely dense. Therefore, Theorem 2 follows from Proposition 4.

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## REFERENCES

- [1] V. Araújo, *Attractors and time averages for random maps*, Annales de l'Institut Henri Poincaré (C) Non Linear Analysis **17** (2000), no. 3, 307–369.
- [2] L. Arnold, *Random dynamical systems*, Springer, 1998.
- [3] V. Baladi, A. Kondah, and B. Schmitt, *Random correlations for small perturbations of expanding maps*, Random Comput. Dynam. **4** (1996), 179–204.
- [4] L. Barreira and J. Schmeling, *Sets of “non-typical” points have full topological entropy and full hausdorff dimension*, Israel Journal of Mathematics **116** (2000), no. 1, 29–70.
- [5] C. Bonatti, L. Díaz, and M. Viana, *Dynamics beyond uniform hyperbolicity: A global geometric and probabilistic perspective*, Springer, 2004.
- [6] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Springer, 1977.
- [7] F. Hofbauer and G. Keller, *Quadratic maps without asymptotic measure*, Communications in mathematical physics **127** (1990), no. 2, 319–337.
- [8] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, 1995.
- [9] S. Kiriki and T. Soma, *Takens’ last problem and existence of non-trivial wandering domains*, arXiv preprint arXiv:1503.06258 (2015).
- [10] D. Ruelle, *Historical behaviour in smooth dynamical systems*, Global analysis of dynamical systems (2001), 63–66.
- [11] F. Takens, *Orbits with historic behaviour, or non-existence of averages*, Nonlinearity **21** (2008), no. 3, T33–T36.

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